NUMBER OF VERTICES IN GELFAND–ZETLIN POLYTOPES

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Abstract. We discuss the problem of counting vertices in Gelfand–Zetlin polytopes. Namely, we deduce a partial differential equation with constant coefficients on the exponential generating function for these numbers. For some particular classes of Gelfand-Zetlin polytopes, the number of vertices can be given by explicit formulas.

1. Introduction and statement of results

Gelfand–Zetlin polytopes play an important role in representation theory [GZ, O] and in algebraic geometry (see [KST]). Let $\lambda_1 \leqslant \ldots \leqslant \lambda_n$ be a non-decreasing finite sequence of integers, i.e. an integer partition. The corresponding Gelfand–Zetlin polytope is a convex polytope in $\mathbb{R}^{\frac{n(n-1)}{2}}$ defined by an explicit set of linear inequalities depending on λ_i. It will be convenient to label the coordinates $u_{i,j}$ in $\mathbb{R}^{\frac{n(n-1)}{2}}$ by pairs of integers (i,j), where i runs from 1 to $n-1$, and j runs from 1 to $n-i$. The inequalities defining the Gelfand–Zetlin polytope can be visualized by the following triangular table.

$$\begin{array}{ccccccccccc}
\lambda_1 & & \lambda_2 & & & \lambda_3 & & \ldots & & \lambda_n \\
& u_{1,1} & & u_{1,2} & & & \ldots & & u_{1,n-1} & \\
& & u_{2,1} & & & \ldots & & u_{2,n-2} & & \\
& & & \ddots & & \ldots & & & & \\
& & & u_{n-2,1} & & & u_{n-2,2} & & & \\
& & & & & u_{n-1,1} & & & &
\end{array} \qquad (GZ)$$

where every triple of numbers a, b, c that appear in the table as vertices of the triangle

$$\begin{array}{ccc} a & & b \\ & c & \end{array}$$

are subject to the inequalities $a \leqslant c \leqslant b$.

In this paper, we discuss generating functions for the number of vertices in Gelfand–Zetlin polytopes. We will use the multiplicative notation for partitions, e.g. $1^{i_1}2^{i_2}3^{i_3}$ will denote the partition consisting of i_1 copies of 1, i_2 copies of 2, and i_3 copies of 3. Given a partition p, we write $GZ(p)$ for the corresponding Gelfand–Zetlin polytope, and $V(p)$ for the number of vertices in $GZ(p)$.

Fix a positive integer k, and consider all partitions of the form $1^{i_1}\ldots k^{i_k}$, where a priori some of the powers i_j may be zero. We let E_k denote the exponential

generating function for the numbers $V(1^{i_1}\dots k^{i_k})$, i.e. the formal power series

$$E_k=\sum_{i_1,\dots,i_k\geqslant 0} V(1^{i_1}\dots k^{i_k})\frac{z_1^{i_1}}{i_1!}\dots\frac{z_k^{i_k}}{i_k!}.$$

Our first result is a partial differential equation on the function E_k:

Theorem 1.1. *The formal power series E_k satisfies the following partial differential equation with constant coefficients:*

$$\left(\frac{\partial^k}{\partial z_1\dots\partial z_k}-\left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial z_2}\right)\dots\left(\frac{\partial}{\partial z_{k-1}}+\frac{\partial}{\partial z_k}\right)\right)E_k=0.$$

E.g. we have

$$E_1(z_1)=e^{z_1},\quad E_2(z_1,z_2)=e^{z_1+z_2}I_0\left(2\sqrt{z_1z_2}\right),$$

where I_0 is the modified Bessel function of the first kind with parameter 0. This function can be defined e.g. by its power expansion

$$I_0(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!^2}.$$

It is also useful to consider ordinary generating functions for the numbers $V(1^{i_1}\dots k^{i_k})$:

$$G_k(y_1,\dots,y_k)=\sum_{i_1,\dots,i_k\geqslant 0} V(1^{i_1}\dots k^{i_k})y_1^{i_1}\dots y_k^{i_k}.$$

We will also deduce equations on G_k. These will be difference equations rather than differential equations. For any power series f in the variables y_1, …, y_k, define the action of the divided difference operator Δ_i on f as

$$\Delta_i(f)=\frac{f-f|_{y_i=0}}{y_i}.$$

Theorem 1.2. *The ordinary generating function G_k satisfies the following equation*

$$(\Delta_1\dots\Delta_k-(\Delta_1+\Delta_2)\dots(\Delta_{k-1}+\Delta_k))\,G_k=0.$$

It is known that the ordinary generating functions G_k can be obtained from exponential generating functions E_k by the Laplace transform. Thus Theorem 1.2 can in principle be deduced from Theorem 1.1 and the properties of the Laplace transform. However, we will give a direct proof.

For $k=1$, 2 and 3, the generating functions G_k can be computed explicitly. It is easy to see that

$$G_1(y_1)=\frac{1}{1-y_1},\quad G_2(y_1,y_2)=\frac{1}{1-y_1-y_2}.$$

We will prove the following theorem:

Theorem 1.3. *The function* $G_3(x,y,z)$ *is equal to*

$$\frac{2xz - y(1-x-z) - y\sqrt{1-2(x+z)+(x-z)^2}}{2(1-x-z)((x+y)(y+z)-y)}.$$

The numbers $V_{k,\ell,m} = V(1^k 2^\ell 3^m)$ can be alternatively expressed as coefficients of certain polynomials:

Theorem 1.4. *The number* $V_{k,\ell,m}$ *for* $k>0$, $\ell>0$, $m>0$ *is equal to the coefficient with* $x^k z^m$ *in the polynomial*

$$\frac{1-xz}{1+xz}\left((1+x)^{k+\ell+m}(1+z)^{k+\ell+m} - (x+z)^{k+\ell+m}\right).$$

Set $s = k+\ell+m$. Note that, since the term $(x+z)^s$ is homogeneous of degree s, the number $V_{k,\ell,m}$, where $k,\ell,m>0$, is also equal to the coefficient with $x^k z^m$ in the power series

$$\frac{(1-xz)(1+x)^s(1+z)^s}{1+xz}.$$

This implies the following explicit formula for the numbers $V_{k,\ell,m}$ $(k,\ell,m>0)$:

$$V_{k,\ell,m} = \binom{s}{k}\binom{s}{m} + 2\sum_{i=1}^{k}(-1)^i\binom{s}{k-i}\binom{s}{m-i}.$$

Note that the sum $\sum_{i=1}^{k}(-1)^i\binom{s}{k-i}\binom{s}{m-i}$ can be expressed as the value of the generalized hypergeometric function ${}_3F_2$, namely, it is equal to $\binom{s}{k-1}\binom{s}{m-1}\,{}_3F_2(1, 1-k, 1-m; 2+\ell+m, 2+k+\ell; -1)$.

2. Recurrence relations

Let R be the polynomial ring in countably many variables x_1, x_2, x_3, ... Define a linear operator $A : R \to R$ by the following formula:

$$A(x_{i_1}x_{i_2}\dots x_{i_k} f) = (x_{i_1}+x_{i_2})(x_{i_2}+x_{i_3})\dots(x_{i_{k-1}}+x_{i_k})f.$$

In this formula, x_{i_1}, ..., x_{i_k} is any finite subset of variables, and $f(x_{i_1}, x_{i_2}, \dots, x_{i_k})$ is any polynomial in these variables. The formula displayed above defines a linear action of A on the entire R. Indeed, any polynomial $P \in R$ can be represented as the sum $P = \sum_S P_S$, where S runs through all finite subsets of variables, and P_S denotes the sum of all terms (monomials together with coefficients) that involve exactly all variables from S and no other variables. The polynomial $A(P_S)$ is defined above, and we extend A by linearity. We also set $A(1) = 1$ by definition.

The operator A thus defined reduces the degrees of all nonconstant polynomials. Therefore, for any polynomial P, there exists a positive integer N such that $A^N(P)$ is a constant, which is independent of the choice of N provided that N is sufficiently large. We let $A^\infty(P)$ denote this constant.

Proposition 2.1. *We have*

$$V(1^{i_1}\dots k^{i_k}) = A^\infty(x_1^{i_1}\dots x_k^{i_k}).$$

Proof. We will argue by induction on the degree $i_1+\dots+i_k$, equivalently, on the dimension of the Gelfand–Zetlin polytope $GZ(1^{i_1}\dots k^{i_k})$. Let π be the linear projection of $GZ(1^{i_1}\dots k^{i_k})$ to the cube C given in coordinates $(u_1,\dots,u_{k-1})$ by the inequalities

$$1\leqslant u_1\leqslant 2\leqslant u_2\leqslant\dots\leqslant k-1\leqslant u_{k-1}\leqslant k. \qquad (C)$$

Namely, we set $u_1=u_{1,i_1}$, $u_2=u_{1,i_1+i_2}$, $\dots$, $u_{k-1}=u_{1,i_1+\dots+i_{k-1}}$. Observe that all vertices of $GZ(p)$ project to vertices of the cube C. Thus it suffices to describe the fibers of the projection π over the vertices of the cube C.

It will be convenient to label the vertices of the cube C by the monomials in the expansion of the polynomial $A(x_1\dots x_k)$. Namely, to fix a vertex of C, one needs to specify, for every j between 1 and $k-1$, which of the two inequalities $j\leqslant u_j$ or $u_j\leqslant j+1$ turns to an equality. Similarly, to fix a monomial in the polynomial $A(x_1\dots x_k)$, one needs to specify, for every j between 1 and $k-1$, which term is taken from the factor (x_j+x_{j+1}), the term x_j or the term x_{j+1}. This description makes the correspondence clear.

Let v be the vertex of the cube C corresponding to a monomial $x_1^{\alpha_1}\dots x_k^{\alpha_k}$. It is not hard to see that the polytope $\pi^{-1}(v)$ is combinatorially equivalent to

$$GZ(1^{i_1-1+\alpha_1}\dots k^{i_k-1+\alpha_k}).$$

Suppose that

$$A(x_1\dots x_k)=\sum_{\alpha_1,\dots,\alpha_k} c_{\alpha_1\dots\alpha_k} x_1^{\alpha_1}\dots x_k^{\alpha_k}.$$

Then we have

$$V(1^{i_1}\dots k^{i_k})=\sum_{\alpha_1,\dots,\alpha_k} c_{\alpha_1\dots\alpha_k} V(1^{i_1-1+\alpha_1}\dots k^{i_k-1+\alpha_k}).$$

Since for any k-tuple of indices α_1, $\dots$, α_k, for which the corresponding coefficient $c_{\alpha_1,\dots,\alpha_k}$ is nonzero, the Gelfand–Zetlin polytope $GZ(1^{i_1-1+\alpha_1}\dots k^{i_k-1+\alpha_k})$ has smaller dimension than $GZ(1^{i_1}\dots k^{i_k})$, we can assume by induction that

$$V(1^{i_1-1+\alpha_1}\dots k^{i_k-1+\alpha_k})=A^\infty(x_1^{i_1-1+\alpha_1}\dots x_k^{i_k-1+\alpha_k}).$$

Hence we have

$$V(1^{i_1}\dots k^{i_k})=\sum_{\alpha_1,\dots,\alpha_k} c_{\alpha_1\dots\alpha_k} A^\infty(x_1^{i_1-1+\alpha_1}\dots x_k^{i_k-1+\alpha_k})=A^\infty(A(x_1^{i_1}\dots x_k^{i_k})).$$

The desired statement follows. □

3. Equations on generating functions E_k and G_k

In this section, we deduce equations on the generating functions E_k and G_k. In particular, we prove Theorems 1.1 and 1.2.

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_k)$, we let z^α denote the monomial $z_1^{\alpha_1} \dots z_k^{\alpha_k}$, and $\alpha!$ denote the product $\alpha_1! \dots \alpha_k!$. The partial derivation with respect to z_ℓ will be written as ∂_ℓ. The power ∂^α will mean $\partial_1^{\alpha_1} \dots \partial_k^{\alpha_k}$. We will write I_ℓ for the operator of integration with respect to the variable z_ℓ. This operator acts on the power series $\sum_{n=0}^\infty a_n z_\ell^n$, where a_n are power series in the other variables, as follows:

$$I_\ell\left(\sum_{n=0}^\infty a_n z_\ell^n\right) = \sum_{n=0}^\infty a_n \frac{z^{\ell+1}}{\ell+1}.$$

We will use the expansion

$$(x_1 + x_2) \dots (x_{k-1} + x_k) = \sum_\alpha c_\alpha x^\alpha,$$

in which the coefficients c_α can be computed explicitly. Let E_k^* be the sum of all terms in E_k divisible by $z_1 \dots z_k$. Then we have (i, j, α being multi-indices of dimension k)

$$E_k^* = \sum_{i>0} A^\infty(x^i)\frac{z^i}{i!} = \sum_{i>0}\sum_\alpha c_\alpha A^\infty(x^{i-1+\alpha})\frac{z^i}{i!} =$$

$$= \sum_\alpha c_\alpha \partial^\alpha I_1 \dots I_k \sum_{i>0} A^\infty(x^{i-1+\alpha})\frac{z^{i-1+\alpha}}{(i-1+\alpha)!} = \sum_\alpha c_\alpha \partial^\alpha I_1 \dots I_k \sum_{j\geqslant\alpha} A^\infty(x^j)\frac{z^j}{j!}.$$

Apply the differential operator $\partial_1 \dots \partial_k$ to both sides of this equation. Note that $\partial_1 \dots \partial_k(E_k^*) = \partial_1 \dots \partial_k(E_k)$. Thus we have

$$\partial_1 \dots \partial_k(E_k) = \sum_\alpha c_\alpha \partial^\alpha \sum_{j\geqslant\alpha} A^\infty(x^j)\frac{z^j}{j!}.$$

Observe also that, since $\alpha \geqslant 0$ whenever $c_\alpha \neq 0$, we have

$$\partial^\alpha \sum_{j\geqslant\alpha} A^\infty(x^j)\frac{z^j}{j!} = \partial^\alpha E_k.$$

This implies Theorem 1.1.

Example: $k=1$ and $k=2$. In the case $k=1$, we have $E_1 = e^{z_1}$. Consider now the case $k=2$. Set $E = E_2$, $x = z_1$ and $y = z_2$. By Theorem 1.1, the function E satisfies the following partial differential equation:

$$E_{xy} = E_x + E_y.$$

This equation can be simplified by setting $E = e^{x+y}u$, then the function u satisfies the equation

$$u_{xy} = u.$$

and the boundary value conditions $u(x,0) = u(0,y) = 1$. We can now look for solutions u that have the form $v(xy)$, where v is some smooth function. This function must satisfy the initial condition $v(0) = 1$ and the ordinary differential equation

$$tv''(t) + v'(t) - v(t) = 0.$$

It is known that the only analytic solution of this initial value problem is $I_0(2\sqrt{t})$, where I_0 is the modified Bessel function of the first kind. Thus $I_0(2\sqrt{xy})$ is a partial solution of the boundary value problem $u_{xy} = u$, $u(x,0) = u(0,y) = 1$. The solution of this boundary value problem is unique (note that the boundary values are defined on characteristic curves!). Therefore, we must conclude that $E(x,y) = e^{x+y}I_0(2\sqrt{xy})$.

The proof of Theorem 1.2 is very similar to the proof of Theorem 1.1. Let G_k^* be the sum of all terms in G_k that are divisible by $y_1 \dots y_k$, i.e.

$$G_k^* = \sum_{i>0} V(1^{i_1} \dots k^{i_k})y^i.$$

Then, similarly to a formula obtained for E_k^*, we have

$$G_k^* = \sum_{\alpha} c_\alpha y_1^{1-\alpha_1} \dots y_k^{1-\alpha_k} \sum_{j \geqslant \alpha} A^\infty(x^j)y^j.$$

Applying the operator $\Delta_1 \dots \Delta_k$ to both sides of this equation, we obtain Theorem 1.2. Similarly to the proof of Theorem 1.1, we need to use that

$$\Delta_1 \dots \Delta_k(G_k^*) = \Delta_1 \dots \Delta_k(G_k)$$

and that

$$\Delta_1^{\alpha_1} \dots \Delta_k^{\alpha_k}(G_k) = y_1^{-\alpha_1} \dots y_k^{-\alpha_k} \sum_{j \geqslant \alpha} A^\infty(x^j)y^j.$$

We will now discuss several examples.

Example: $k = 1$ and $k = 2$. For $k = 1$, we have the following equation: $\Delta_1 G_1 = G_1$, i.e. $G_1(y_1) - G_1(0) = y_1 G_1(y_1)$. Knowing that $G_1(0) = 1$, this gives

$$G_1(y_1) = 1 + y_1 + y_1^2 + \dots = \frac{1}{1 - y_1}.$$

Suppose that $k = 2$. Set $G = G_2$, $x = y_1$, $y = y_2$. The function G satisfies the following equation

$$\Delta_x \Delta_y G = \Delta_x G + \Delta_y G.$$

Note that $G(x,0) = G_1(x)$ and $G(0,y) = G_1(y)$. Therefore, the right-hand side can be rewritten as

$$\frac{G - \frac{1}{1-y}}{x} + \frac{G - \frac{1}{1-x}}{y}.$$

The left-hand side is

$$\Delta_x \left(\frac{G - \frac{1}{1-x}}{y} \right) = \frac{1}{x} \left(\frac{G - \frac{1}{1-x}}{y} - \frac{\frac{1}{1-y} - 1}{y} \right).$$

Solving the linear equation on G thus obtained, we conclude that

$$G = \frac{1}{1-x-y}.$$

Example: $k=3$. We set $G = G_3$, $x = y_1$, $y = y_2$ and $z = y_3$. The function G satisfies the following equation: $\Delta_x\Delta_y\Delta_z G = (\Delta_x + \Delta_y)(\Delta_y + \Delta_z)G$. This equation can be rewritten as follows:

$$\Delta_y^2 G = \frac{G(1-x-y-z)-1}{xyz}.$$

Suppose that $G = G(x,0,z) + A(x,z)y + \dots$, where dots denote the terms divisible by y^2. Then we have

$$\Delta_y^2 G = G - G(x,0,z) - A(x,z)y = G - \frac{1}{1-x-z} - A(x,z)y.$$

Substituting this into the equation, we can solve the equation for G in terms of A:

$$G = \frac{-xz + y(1-x-z)(1-A(x,z)xz)}{(1-x-z)\left(y-(x+y)(y+z)\right)}.$$

Since the power series $1-x-z$ is invertible, it follows that G has the form

$$\frac{a+by}{y-(x+y)(y+z)},$$

where a and b are some power series in x and z. Let λ and μ be the two solutions of the equation $y = (x+y)(y+z)$, namely,

$$\lambda, \mu = \frac{1-x-z \pm \sqrt{1-2(x+z)+(x-z)^2}}{2}.$$

The signs are chosen so that, at the point $x = z = 0$, we have $\lambda = 1$ and $\mu = 0$. Then

$$\frac{1}{y-(x+y)(y+z))} = \frac{c}{y-\lambda} + \frac{d}{y-\mu},$$

where c and d are some power series in x and z. Note that, since $(y-\lambda)^{-1}$ makes sense as a power series, $c(a+by)/(y-\lambda)$ can be represented as a power series in x, y and z. Thus the function $d(a+by)/(y-\mu)$ must also be representable as a power series in x, y and z. However, this is only possible if the numerator is a multiple of the denominator, i.e. $(a+by) = e(y-\mu)$, where the coefficient e is a power series of x and z. It follows that G is equal to $e(y-\lambda)^{-1}$. The coefficient e can be found from the condition $G(x,0,z) = \frac{1}{1-x-z}$:

$$G = \frac{1}{1-x-z}\frac{\lambda}{\lambda-y} =$$

$$= \frac{2xz - y(1-x-z) - y\sqrt{1-2(x+z)+(x-z)^2}}{2(1-x-z)((x+y)(y+z)-y)}.$$

1	
1	1

1		
2	2	
1	2	1

1			
3	3		
3	7	3	
1	3	3	1

1				
4	4			
6	16	6		
4	14	16	4	
1	4	6	4	1

1					
5	5				
10	30	10			
10	40	52	10		
5	23	40	30	5	
1	5	10	10	5	1

FIGURE 1. Triangular tables T^s containing the numbers $V^s_{k,m}$. Southwest corners of these tables are located at $(0, 0)$.

4. Proof of Theorem 1.4

In this section, we will prove Theorem 1.4, which expresses the numbers $V_{k,\ell,m}$ as coefficients of certain Laurent polynomials. The numbers $V_{k,\ell,m}$ satisfy the following recurrence relation:

$$V_{k,\ell,m} = V_{k-1,\ell,m} + V_{k,\ell-1,m} + V_{k,\ell,m-1} + V_{k-1,\ell+1,m-1}$$

provided that k, ℓ, $m > 0$, and the following initial conditions:

$$V_{0,\ell,m} = V_{\ell,m}, \quad V_{k,0,m} = V_{k,m}, \quad V_{k,\ell,0} = V_{k,\ell}.$$

Set $V^s_{k,m} = V_{k,s-k-m,m}$. Then we can write the following recurrence relations on the numbers $V^s_{k,m}$:

$$V^s_{k,m} = V^{s-1}_{k-1,m} + V^{s-1}_{k,m-1} + V^{s-1}_{k-1,m-1} + V^{s-1}_{k,m}$$

provided that $k \geqslant 1$, $m \geqslant 1$, $k + m \leqslant s - 1$, and

$$V^s_{k,m} = V^{s-1}_{k-1,m} + V^{s-1}_{k,m-1} + V^{s-1}_{k-1,m-1}$$

provided that $k + m = s$.

For a fixed s, we can arrange the numbers $V^s_{k,m}$ into a triangular table T^s of size s as shown on Figure 1. Namely, the number $V^s_{k,m}$ is placed into the cell, whose southwest (lower left) corner is at position (k, m). The next table T^{s+1} can be obtained from the table T^s as follows. First, we add to every element of T^s its south, west and southwest neighbors. Next, we add a line of cells, whose positions (k, m) satisfy the equality $k + m = s$. In every cell of this line, we put the sum of the south and west neighbors. Note that, by construction, the boundary of every table T^s consists of binomial coefficients.

Consider the generating function $G = G_3$ for the numbers $V_{k,\ell,m}$. The splitting of G into homogeneous components can be obtained by expanding the function $G(xy, y, zy)$ into powers of y. We set

$$G(xy, y, zy) = \sum_{s=0}^{\infty} g_s(x, z) y^s$$

$$
\begin{array}{|c|c|}\hline 0 & -1 \\ \hline 1 & 0 \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|}\hline 0 & -2 & -1 \\ \hline 2 & 0 & -2 \\ \hline 1 & 2 & 0 \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|c|}\hline 0 & -3 & -3 & -1 \\ \hline 3 & 0 & -7 & -3 \\ \hline 3 & 7 & 0 & -3 \\ \hline 1 & 3 & 3 & 0 \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|c|c|}\hline 0 & -4 & -6 & -4 & -1 \\ \hline 4 & 0 & -16 & -14 & -4 \\ \hline 6 & 16 & 0 & -16 & -6 \\ \hline 4 & 14 & 16 & 0 & -4 \\ \hline 1 & 4 & 6 & 4 & 0 \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|c|c|c|}\hline 0 & -5 & -10 & -10 & -5 & -1 \\ \hline 5 & 0 & -30 & -40 & -23 & -5 \\ \hline 10 & 30 & 0 & -52 & -40 & -10 \\ \hline 10 & 40 & 52 & 0 & -30 & -10 \\ \hline 5 & 23 & 40 & 30 & 0 & -5 \\ \hline 1 & 5 & 10 & 10 & 5 & 0 \\ \hline \end{array}
$$

FIGURE 2. The skew-symmetric tables $\tilde{T}^s$.

$$
\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & -1 & 0 \\ \hline 1 & 0 & 0 \\ \hline \end{array}
\xrightarrow{1}
\begin{array}{|c|c|c|}\hline 0 & -1 & -1 \\ \hline 1 & 0 & -1 \\ \hline 1 & 1 & 0 \\ \hline \end{array}
\xrightarrow{2}
\begin{array}{|c|c|c|}\hline 0 & -2 & -1 \\ \hline 2 & 0 & -2 \\ \hline 1 & 2 & 0 \\ \hline \end{array}
$$

$$
\begin{array}{|c|c|c|c|}\hline 0 & 0 & 0 & 0 \\ \hline 0 & -2 & -1 & 0 \\ \hline 2 & 0 & -2 & 0 \\ \hline 1 & 2 & 0 & 0 \\ \hline \end{array}
\xrightarrow{1}
\begin{array}{|c|c|c|c|}\hline 0 & -2 & -3 & -1 \\ \hline 2 & 0 & -5 & -3 \\ \hline 3 & 5 & 0 & -2 \\ \hline 1 & 3 & 2 & 0 \\ \hline \end{array}
\xrightarrow{2}
\begin{array}{|c|c|c|c|}\hline 0 & -3 & -3 & -1 \\ \hline 3 & 0 & -7 & -3 \\ \hline 3 & 7 & 0 & -3 \\ \hline 1 & 3 & 3 & 0 \\ \hline \end{array}
$$

FIGURE 3. The rules of generating the tables $\tilde{T}^s$.

Then we have

$$g_s(x,z) = \sum_{k=0}^{s} \sum_{m=0}^{s-k} V_{k,m}^s x^k z^m.$$

Thus the coefficients of the polynomial g_s are precisely elements of the table T^s. The recurrence relations on the numbers $V_{k,m}^s$ displayed above imply the following property of the generating functions g_s:

Proposition 4.1. *The polynomials g_s satisfy the following recurrence relations:*

$$g_{s+1} = (1+x+z)g_s + \tau_{\leqslant s}(xzg_s),$$

where the truncation operator $\tau_{\leqslant s}$ acts on a polynomial by removing all terms, whose degrees exceed s.

Consider the polynomials

$$h_s(x,z) = g_s(x,z) - (xz)^s g_s(z^{-1}, x^{-1}).$$

Geometrically, these polynomials can be described as follows. Let $\tilde{T}^s$ denote the table, into which we put all coefficients of the polynomial h_s, see Figure 2. The lower left triangle of size $s-1$ is the same in the tables T^s and $\tilde{T}^s$. The table $\tilde{T}^s$ is skew-symmetric with respect to the main diagonal. These two properties give a unique characterization of the tables $\tilde{T}^s$.

The rules, by which the tables $\tilde{T}^s$ are formed, are the following (see Figure 3). The first table $\tilde{T}^1$ is by definition the left-most table shown on Figure 2. The next table $\tilde{T}^{s+1}$ is obtained inductively from the preceding table $\tilde{T}^s$ in two steps. In the first step, we add to every element of $\tilde{T}^s$ its immediate west, south and southwest neighbors. In the second step, we modify elements in two diagonals of the table, namely, the elements, whose positions (mesured by southwest corners) (k, m) satisfy the equality $k+m = s$ or $k+m = s+2$. To the cell at position (k, m), where $k + m = s$, we add the binomial coefficient $\binom{k+m}{m}$. From the cell at position $(k + 1, m + 1)$, we subtract this binomial coefficient.

We have the following recurrence relation on the polynomials h_s:

$$h_{s+1} = h_s(1 + x)(1 + z) + (1 - xz)(x + z)^s,$$

which does not contain truncation operators. Therefore, the generating function $H = \sum_{s=0}^{\infty} h_s y^s$ satisfies the following linear equation:

$$H = 1 + y((1 + x)(1 + z)H + (1 - xz)(1 - y(x + z))^{-1}).$$

Solving this equation, we find that

$$H = \frac{y(1 - xz)}{(1 - y(x + z))(1 - y(1 + x)(1 + z))}.$$

Knowing the generating function H, we can now obtain an explicit formula for the polynomials h_s, namely,

$$h_s(x, z) = \frac{1 - xz}{1 + xz}\left((1 + x)^s(1 + z)^s - (x + z)^s\right).$$

Theorem 1.4 is thus proved.

Open problems.

(1) Prove or disprove: the generating function G_4 is algebraic. Note that G_1 and G_2 are rational, and G_3 is algebraic.
(2) Deduce differential or difference equations on the generating functions for the f-vectors and for the modified h-vectors of Gelfand–Zetlin polytopes.

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